Maxima & Minima

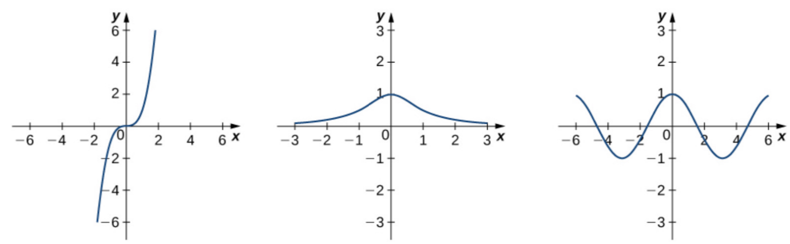
Absolute Maxima or Minima:

Let $f: I \to \mathbb{R}$ (reals) be a function. Let $c \in I$.

f has $\boxed{\text{absolute maxima}}$ if $f(c) \geq f(x)$ for all $x \in I$.

f has $\boxed{\text{absolute minima}}$ if $f(c) \leq f(x)$ for all $x \in I$.

absolute extrema at $x = c$



$f(x) = x^3$ on $(-\infty, \infty)$
No absolute maximum
No absolute minimum

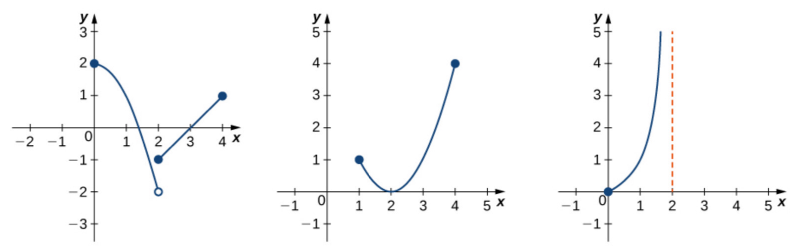
(a)

$f(x) = \frac{1}{x^2+1}$ on $(-\infty, \infty)$
Absolute maximum of 1 at $x = 0$
No absolute minimum

(b)

$f(x) = \cos(x)$ on $(-\infty, \infty)$
Absolute maximum of 1 at $x = 0$, $\pm 2\pi$, $\pm 4\pi$...
Absolute minimum of -1 at $x = \pm\pi$, $\pm 3\pi$...

(c)



$f(x) = \begin{cases} 2 - x^2 & 0 \leq x < 2 \\ x - 3 & 2 \leq x \leq 4 \end{cases}$
Absolute maximum of 2 at $x = 0$
No absolute minimum

(d)

$f(x) = (x-2)^2$ on $[1, 4]$
Absolute maximum of 4 at $x = 4$
Absolute minimum of 0 at $x = 2$

(e)

$f(x) = \frac{x}{2-x}$ on $[0, 2)$
No absolute maximum
Absolute minimum of 0 at $x = 0$

(f)

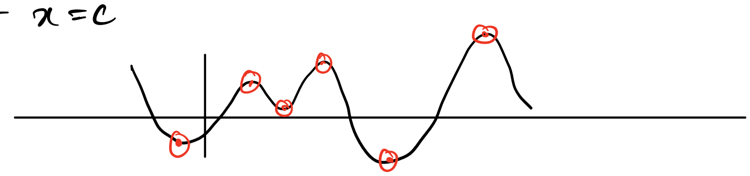
Local Maxima & Minima:

Let $f: I \to \mathbb{R}$ be a function. Let J be a subset of I such that c is in J.

f has $\boxed{\text{local maxima}}$ if $f(c) \geq f(x)$ for all $x \in J$.

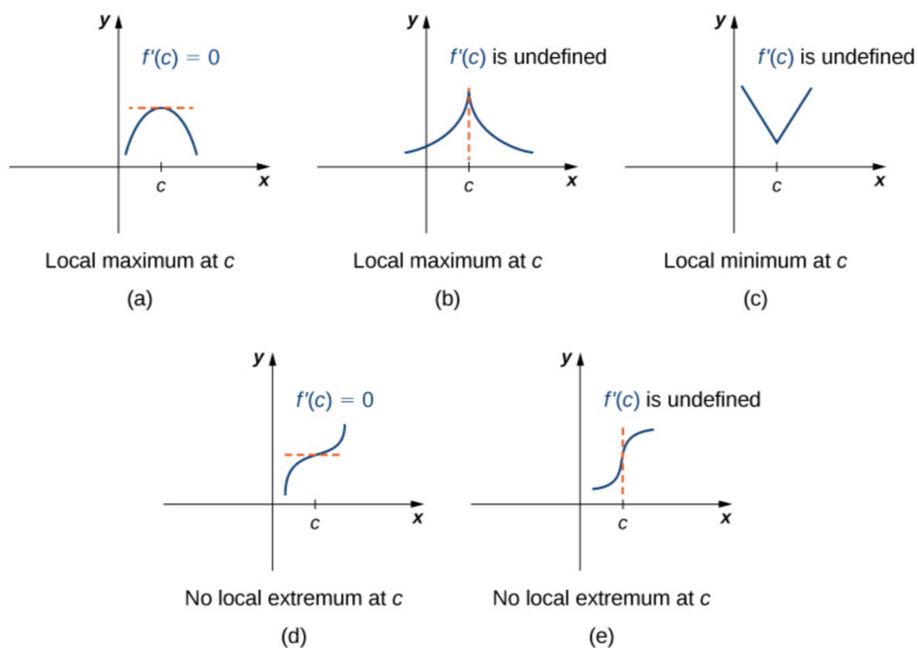
f has $\boxed{\text{local minima}}$ if $f(c) \leq f(x)$ for all $x \in J$.

local extrema at $x = c$



Critical Point : f has a critical point at $x=c$ if $f'(c)=0$ or $f'(c)$ undefined.

Fermat's Theorem : A function f has local extremum at $x=c$ if f is differentiable at $x=c$ & $f'(c)=0$.



Eg. $f(x) = -x^2+3x-2$, $x \in [1,3]$

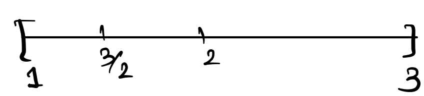
↳ Note, this is a closed interval. So, we need to examine the end points.

For extrema: $f'(x)=0$ or undefined

Now, $f'(x) = -2x+3$

$\Rightarrow -2x+3=0 \rightsquigarrow 2x=3 \rightsquigarrow x=\frac{3}{2}$

At $x=\frac{3}{2}$, $\boxed{f(x)=0.25}$ → Absolute Maxima



$x=1$, $f(x)=0$ ⇝ Local extrema

$x=3$, $\boxed{f(x)=-2}$ → Absolute Minima

Detection of Local Extrema:

Say $x=c$ is a critical point of f, ie, $f'(c)=0$.

<u>Case 1</u>: $f'(x)$ is +ve for $x<c$ & $f'(x)$ is −ve for $x>c$

$\Rightarrow f(c)$ is a local maxima.

<u>Case 2</u>: $f'(x)$ is −ve for $x<c$ & $f'(x)$ is +ve for $x>c$.

$\Rightarrow f(c)$ is a local minima.

<u>Case 3</u>: $f'(x)$ has the same sign on both side of $x=c$

$\Rightarrow f(c)$ is $\underbrace{\text{neither a local maxima or minima}}_{\text{saddle point}}$

2nd derivative test:

Say $x=c$ is a critical point of f, ie, $f'(c)=0$.

<u>Case 1</u>: $f''(c)>0 \Rightarrow f(c)$ is a local minima.

<u>Case 2</u>: $f''(c)<0 \Rightarrow f(c)$ is a local maxima.

<u>Case 3</u>: $f''(c)=0 \Rightarrow$ Inconclusive.